Swami Ramanand Teerth Marathwada University, Nanded Maharashtra, India.

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"A Study of the Existence & Uniqueness of Some Classes of Differential Equations with Different Types of Fractional Derivatives and Their Applications"

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A Study of the Existence & Uniqueness of Some Classes of Differential Equations with Different Types of Fractional Derivatives and Their Applications

Introduction:

Fractional calculus is a mathematical branch investigating the properties of derivatives and integrals of non-integer orders (called fractional derivatives and integrals, briefly differintegrals). In particular, this discipline involves the notion and methods of solving of differential equations involving fractional derivatives of the unknown function (called fractional differential equations). The history of fractional calculus started almost at the same time when classical calculus was established. It was first mentioned in Leibniz's letter to L'Hospital in 1695, where the idea of semiderivative was suggested. During time fractional calculus was built on formal foundations by many famous mathematicians, e.g. Liouville, Grunwald, Riemann, Euler, Lagrange, Heaviside, Fourier, Abel etc. A lot of them proposed original approaches, which can be found chronologically in [10]. The theory of fractional calculus includes even complex orders of differintegrals and left and right differintegrals (analogously to left and right derivatives). The fact, that the differintegral is an operator which includes both integer-order derivatives and integrals as special cases, is the reason why in present fractional calculus becomes very popular and many applications arise. The fractional integral may be used e.g. for better describing the cumulation of some quantity, when the order of integration is unknown, it can be determined as a parameter of a regression model as Podlubny presents in [1]. Analogously the fractional derivative is sometimes used for describing damping. Other applications occur in the following fields: fluid flow, viscoelasticity, control theory of dynamical systems, diffusive transport akin to diffusion, electrical networks, probability and statistics, dynamical processes in self-similar and porous structures, electrochemistry of corrosion, optics and signal processing, rheology etc

There are two basic definitions of fractional derivative [1, 2] which are

a) Riemann Liouville Fractional Derivative:

For $\alpha \in [n-1, n)$, the α-derivative of f is

$$D_a^{\alpha}(f)(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^{\alpha} \frac{f(x)}{(t-x)^{(\alpha-n+1)}} dx.$$

b) Caputo's Fractional Derivative:

For $\alpha \in [n-1, n)$, the α-derivativeof f is

$$D_a^{\alpha}(f)(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^{\alpha} \frac{f(x)}{(t-x)^{(\alpha-n+1)}} dx.$$

The Laplace transformation method can be used to solve Fractional Differential Equations [1, 3, 4]. The Laplace transform of Fractional derivative is given by,

a) Laplace Transform of Riemann Liouville Fractional Derivative

$$L\left\{{}_0^{a}D_t^{p} f(t); s\right\} = s^p F(s) - \sum_{k=0}^{n-1} s^k \left[{}_0^{a}D_t^{p-k-1} f(t)\right]_{t=0},$$

where $(n-1 < p \leq n)$

b) Laplace Transform of Caputo Fractional Derivative

$$L\left\{{}_0^{c}D_t^{p} f(t); s\right\} = s^p F(s) - \sum_{k=0}^{n-1} s^{p-k-1} f^{(k)}(0),$$

where $(n-1 < p \leq n)$.

Review of literature and recent developments in the subject area:

In 2009, Yong Zhou [5], had discussed initial value problem for a system of fractional differential equations and new criteria on existence and uniqueness of solutions are obtained. In 2010, Jun Wu [6], shows some new existence and uniqueness results for the fractional differential equations with delay and impulse effects. In 2010, Jiqin Deng [7], by using the fixed point theory, he studied the

existence and uniqueness of initial value problems for nonlinear fractional differential equations and obtain a new result.

In 2011, LinLi Lv [8], studied the existence and uniqueness of fractional differential equations with boundary value conditions. A new generalized singular type Gronwall inequality is given to obtain important a priori bounds. In 2012, Shayma Adil Murad [9], have investigate the existence and uniqueness solutions of the nonlinear fractional differential equation of an arbitrary order with integral boundary condition. The result is an application of the Schauder fixed point theorem and the Banach contraction principle. In 2013, Ahmed Anber et al. [10], studied a class of boundary value problems of nonlin-ear fractional differential equations with integral boundary condition. In 2014, Udita N. Katugampola [11], introduced a new fractional derivative which generalizes two familiar fractional derivatives, namely, the Riemann-Liouville and the Hadamard fractional derivatives to a single form.

In 2014, J. A. Nanware et al. [12], they applied the comparison result without locally H¨older continuity due to Vasundhara Devi to develop monotone iterative method for the problem and obtain existence and uniqueness of a solution of the problem. In 2015, Mohamed I. Abbas [13], was proved the existence and uniqueness of solution for fractional differential equations with Riemann-Liouville fractional integral boundary conditions. The first existence and uniqueness result is based on Banach's contraction principle. Moreover, other existence results are also obtained by using the Krasnoselskii fixed point theorem. In 2016, Haiyan Zhang et al. [14], they investigated the existence and uniqueness of solutions for a coupled system of nonlinear fractional differential equations with Riemann-Liouville fractional integral boundary conditions. By applying a variety of fixed point theorems, combining with a new inequality of fractional order form, some sufficient conditions are established.

In 2017, Yufeng Sun et al. [15], studied the existence and uniqueness of solutions for a class of boundary value problems of nonlinear fractional order differential equations involving the Caputo fractional derivative by employing the Banach's contraction principle and the Schauder's fixed point theorem. In 2018, Y.Y. Gambo et al. [16], studied Cauchy problems for differential equations with the derivative in the space of continuously differentiable functions. Using Banach fixed

point theorem, the existence and uniqueness of solution to the considered Cauchy problem is proved. In 2019, Jiahui An et al. [17], by using the Banach contraction mapping principle, they studied the existence and uniqueness of solutions for initial value problem of fractional differential equations of variable-order.

In 2021, Boulfoul, A., et al. [18], studied concerned with the existence and uniqueness of solutions for nonlinear fractional integro-differential equation on an unbounded domain in a weighted Banach space. New results are showed owing to the fixed point theorems of Banach and Krasnoselskii. Illustrate the results by examples to ending the theoretical study. In 2022, Sachin Kumar Verma et al. [19], they considered a boundary value problem of differential equations of fractional order involving the nonlocal boundary condition. they established sufficient conditions for the existence of solution of the boundary value problem with the help of Schaefer's fixed point theorem.

In 2022, Abdelkrim Salim and Mouffak Benchohra [20], presented some results on existence and uniqueness for a class of boundary value problems for iterative fractional differential equations with generalized Caputo fractional derivative. In 2022, Vidushi Gupta et al. [21], have studied the investigation of existence and uniqueness of impulsive dynamical fractional systems with quadratic perturbation of second type subject to non-local boundary conditions is presented and proved. By employing the fractional theory, Banach contraction technique, and Krasnoselskii's fixed point theorem, they derived some sufficient conditions to ensure the existence of the system.

In 2023, El Mfadel, Ali, Said Melliani, and M'hamed Elomari [22], they have established some new sufficient conditions for the existence of solutions to a class of boundary value problem for nonlinear fractional differential equations with Ψ-Caputo fractional derivatives of order $\beta \in (2, 3)$.

Objectives of proposed research:

1) To study basic concepts of calculus.

2) To study some methods to solve differential equations.

3) To study fractional differential equation.

4) To study and analysis of development in results for fractional differential equation.

5) To study and derived some results on an existence a problem and uniqueness of different fractional initial value problem.

6) The proposed work is to prove that, how mathematics is useful in other branches of science. We hope that our research work will be useful in applied sciences and engineering.

Research Methodology:

The following steps will be including in the methodology:

1) To refer the relevant reference books on fractional differential equations.

2) To study Mathematica, MATLAB, Python, Mathcad software's.

3) To study Krasnoselskii's fixed point theorem and the Banach contraction mapping principle.

4) To study the existence and uniqueness of fractional initial value problems.

5) To study the existence and uniqueness of solutions for a class of initial value problems of nonlinear fractional order differential equations involving the Riemann Liouville and Caputo fractional derivative.

6) To represents existence and uniqueness results for different fractional differential equations.

Importance of study:

Fractional calculus is becoming a useful and key tool in the analysis of scientific problems in Physics, Engineering, biology and Economics, Fractional formalism to be useful in solving practical problems, Fractional calculus, Integration and Differentiation of an arbitrary or fractional order, provides new tools that expand the descriptive power of calculus beyond the familiar integer-order concept of rate of change and area under the curve.

International Status and National Status:

Many scientists and researchers many more contributed in the development of fractional differential equations and Fractional calculus like, Leibtniz, Euler, Lagrange, Laplace, Fourier, Abel, Liouville, Grunwald, Riemann, Riesz, Miller, Ross, Igor Prodlubny, Richtmeyer, Morton, Smith, Anderson, Roache, Lynn R. Willam , Nguyen Huu Denn, Yong Zhou, Jun Wu, Mohamed I. Abbas, Abdelkrim Salim And Mouffak Benchohra, El Mfadel, Ali, Said Melliani, and M'hamed Elomar

Indian's also contributed remarkable in the development of fractional calculus like R. K. Sexana, V.Srivastava, K. N. Rai, S. Das, K. Gupta, S. Saha Roy, V. B. L. Chawrasia, D. Kumar, Pradeep Kumar, D. Bahuguna, Aalam, Suneel Kumar, Sunny Chauhan and Pant. S. P. Acharya worked on some results on fixed point in uniform space. S.L. Sing from Haridwar, R. P. Pant from Nainital, D.N Chaliszar from Gujrat, and many more from India had worked on fixed point theorems and its various applications. Regional Status: Many researchers from Maharashtra have been actively involved in the study of fractional calculus. Like M. Kolwankar, A. K. Pani, PSV Nataraj, S. Bhalekar, V. Daftardar- Gejji, J. Verma, D. B. Dhaigude, obtained common fixed point theorems in Dynamic programming.

Scope of research work:

1) Although a number of researchers have developed new operators involving several special functions, study of their properties, introduce alternate definitions and generalizations of these operators but still there are several functions like I-function, Aleph function, Mittag-Leffler function etc.

2) Even many Transformations and their applications has been developed by researchers for solving many problems of science and technology, but an easy, suitable and number of alternate approaches is desired to solve different problems of different domains, so there is a good scope of development in this area.

3)There are several fractional initial value problems with unknowable solutions, and even if they do, there is no guarantee of their uniqueness.

4)There are several fractional boundary value problems with unknowable solutions, and even if they do, there is no guarantee of their uniqueness.

Expected outcomes:

1. We will try to analyse the development in fractional differential equation and fractional initial value problem.
2. We will try to derive existence and uniqueness results for different fractional differential equations.
3. We will try to derive existence and uniqueness results for different fractional initial value problem.
4. We will try to qualitative analyse the solution of different fractional initial value problem by various methods.
5. To find the approximate solutions of fractional differential equations by various Mathematical software.
6. Try to publish our research work in reputed journal.

Purpose of carrying out proposed research work:

The purpose of the research is to develop new results on existence and uniqueness of different initial value problem. The solution of fractional differential equations will be obtained by various methods and compared using mathematical software. Also we will study existence and uniqueness results for different fractional initial value problem and numerical methods to obtain the solutions of fractional partial initial value problem. We will compare the solutions obtain by Analytical methods and Numerical methods. The results will be represented graphically.

Plan of work:

First Year Plan:

1) To refer the reference books on differential equation and fractional differential equation.

2) To investigate the number of reputed research papers in fractional calculus.

3) To study software's like Python, Mathematica and Matlab.

Second Year Plan:

1)To study fractional calculus.

2)To study fractional differential equations.

3)To obtain the solution of initial and boundary value problems of fractional differential equation.

4)To study existence and uniqueness of different fractional differential equations.

Third Year Plan:

1)To develop some results on existence and uniqueness of different fractional differential equation.

2)To publish our work in reputed journals.

3)Submission of completed work to concerned authority.

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